ETMAG

CORONALECTURE 7

Vector spaces

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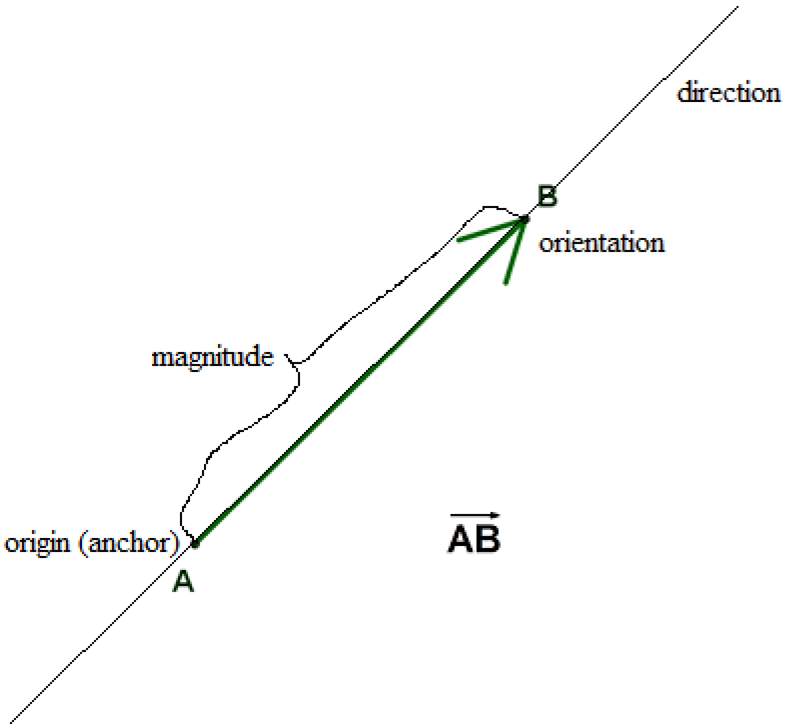
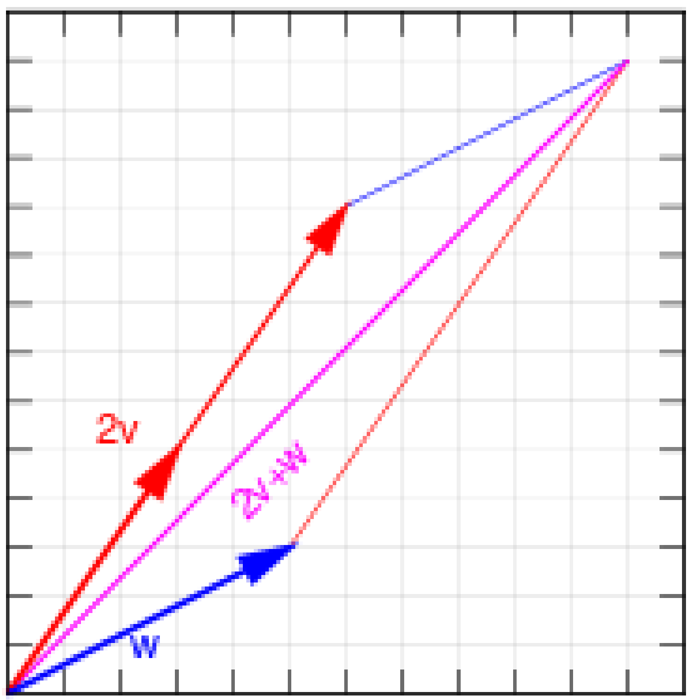
Introduction.

Vectors often appear in physics where they are used to represent quantities such as a force, the velocity or acceleration of an object and others, who are not fully representable by a single number like, for example, the mass of an object or the volume of a solid. The fact that they are characterized by such properties as the magnitude, the direction, orientation and, often, a point of application (as in the case of a force) suggests that they may be represented as arrows whose length is proportional to the magnitude. The other attributes like direction, orientation and the anchor point are more or less self-explanatory.

Geometrically we could identify a vector with an ordered pair of points, one point being the *anchor point* or the *origin* while the location the other one depends on the remaining attributes of the quantity which is being represented by the vector.



Two vectors sharing the anchor point can be added using the parallelogram rule. A vector can be *scaled* by a number, a *scalar*. Scaling preserves the point of origin and the direction of the vector. It may affect the orientation (if the scalar is negative) and the length (if the scalar is different from both 1 and –1). Hence, in order to create the algebra of vectors we consider only vectors anchored at a chosen single point called the origin.



In order to use algebraic approach to vectors we consider the space $\mathbf{R}^2$ or $\mathbf{R}^3$ or some such and we assume that all vectors originate at (0, … ,0). Thus every vector is uniquely identified by its endpoint. This strategy results in a very easy algebraic definition of vector operations. Namely, if you have vectors v_1 and v_2 represented by endpoints (a,b) and (c,d) then v_1+v_2 is represented by (a+c,b+d) and p(a,b) by (pa,pb)

Often we write (a,b) + (c,d) = (a+c,b+d) and p(a,b) = (pa,pb) but you should be aware that this does not mean that we add or scale points of the plane (or other Euclidean space). We add and scale vectors who by default both originate at (0, …,0) and terminate at (a,b) and (c,d) respectively.

Definition. (of a vector space)

A *vector space* (also called *linear space*) is an ordered triple (V, K, *f*) where

- V is an Abelian group with operation usually denoted by +, whose elements are called *vectors*
- K is a field with operations denoted, somewhat confusingly, also by + and ·, whose element are called *scalars*
- *f* is a function from K×V into V called *scaling*. *f*(*p*,v) is often, confusingly, denoted by *p*·v

Such that

1. $(\forall\lambda\in K)(\forall u, v\in V)\ \lambda\cdot(u + v) = \lambda\cdot v + \lambda\cdot v$
2. $(\forall\alpha, \beta\in K)(\forall v\in V)\ (\alpha + \beta)\cdot v = \alpha\cdot v + \beta\cdot v$
3. $(\forall\alpha, \beta\in K)(\forall v\in V)\ (\alpha\cdot\beta)\cdot v = \alpha\cdot(\beta\cdot v)$
4. $(\forall v\in V)\ 1\cdot v = v$, where 1 denotes the identity of field multiplication.

Notice the ambiguity caused by the double meaning of + and · symbols. This is a BAD, UGLY monster but it is traditional. We let context decide which "+" means scalar addition and which – vector addition. Otherwise we would have to introduce extra symbols for scaling and vector addition which would also confuse some people. And would be hard to type.

Switching to the old presentation for some examples.

Example. (A REALLY outlandish one)

Let X be any set. We will use V = $(2^X, \div)$ as the Abelian group of vectors, where $\div$ denotes the operation of symmetric difference of sets, $A \div B = (A \cup B) \setminus (A \cap B)$. We will also use $(\mathbb{Z}_2, \oplus, \otimes)$ as the field of scalars. Scaling is defined as follows:

for every set A, $0 \cdot A = \varnothing$ and $1 \cdot A = A$.

Comprehension.
Check that the above structure is a vector space.

FAQ. 1

What the hell is a vector?
The only proper answer to this question, even though a little confusing, is *"A vector is an element of a vector space"*. The previous example teaches us that sets can be vectors. In other examples we have seen numbers, complex numbers, n-tuples of numbers, functions, polynomials etc. playing the role of vectors.

FAQ. 2
What the hell is a scalar then?

Well, you probably realize that the answer will equally trivial (or disturbing). An (every, really) element of a field may be called a scalar if somebody decides to construct a vector space using this particular field as the second element of the ordered triple constituting a vector space.

Example.
In the vector space of real numbers over the field of real numbers the numbers are both vectors and scalars.

In $\mathbb{C}$ over $\mathbb{R}$ complex numbers are vectors, real numbers are scalars.

In 2^X over $\mathbb{Z}_2$ vectors are subsets of X and there are but two scalars, 0 and 1.

That's what makes general study of vector spaces useful. Whatever facts we discover they will be true in each and every of these spaces.

Theorem. (Arithmetic properties of vector spaces)
In every vector space V over a field K

1. for every vector v, $0v=\theta$, θ represents the zero vector.
2. for every scalar p, $p\theta = \theta$.
3. for every scalar p and vector v, $(-p)v = p(-v) = -(pv)$.
4. for every scalar p and vector v, $pv = \theta$ implies $p=0$ or $v= \theta$.

Proof left as a comprehension exercise.

Examples. (of subspaces or not-subspaces)

Decide which subsets are subspaces:

1. $\{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
2. $\{(x, y) \in \mathbb{R}^2 : x + y \geq 0\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
3. $\{(x, y) \in \mathbb{R}^2 : x = 5y\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
4. $\{(x, y) \in \mathbb{R}^2 : x^2 = y\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
5. $\{(x, y, z) \in \mathbb{R}^3 : x + y - 3z = 1\}$ in $\mathbb{R}^3$ over $\mathbb{R}$
6. $\left\{\{a, b\}, \{a\}, \emptyset\right\}$ in $2^{\{a,b,c\}}$ over $\mathbb{Z}_2$